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On Pro-p Groups Admitting a Fixed-Point-Free Automorphism

Pavel Shumyatsky[1]

TO SAID SIDKI ON HIS 60TH BIRTHDAY

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1. INTRODUCTION

An automorphism ϕ of a group G is called fixed-point-free (FPF for short) if the subgroup of fixed points $C_G(\phi) = \{x \in G; x^{\phi} = x\}$ is trivial. It is well-known that existence of an FPF automorphism ϕ of a finite group G has a strong impact on the structure of G. For example, if ϕ is of prime order q then G is nilpotent [15] and the nilpotency class of G is bounded by some function of q [6]. The minimal function $h(q)$ with the above property is called the Higman function. Also, it follows from the classification of finite simple groups that if a finite group G admits an FPF automorphism ϕ such that $(|G|, |\phi|) = 1$ then G is soluble [3, 1.48]. Moreover, by a result of Th. Berger [1] the Fitting height of G is bounded by the number of prime divisors of $|\phi|$, counting multiplicities.

However, the structure of finite p-groups admitting an FPF automorphism ϕ of arbitrary given order is still a mystery. An unsolved problem is whether the derived length of such groups is bounded by some function of $|\phi|$. Although there is varied evidence suggesting that this is so (for instance, by a theorem of V. A. Kreknin [10] the derived length of any Lie algebra admitting an FPF automorphism of order n is at most $2^{n-1} - 1$), the problem remains unapproachable.

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In this paper we examine the structure of pro-p groups admitting an FPF automorphism of finite order. Throughout the paper automorphisms of topological groups will be assumed to be continuous without explicitly stating this. Note that any (abstract) automorphism of a finitely generated pro-p group is continuous [2, p. 32].

THEOREM A. *Let p be a prime, and G a finitely generated pro-p group admitting an FPF automorphism ϕ of finite order n relatively prime to p. Assume that for any element $\psi \in \langle \phi \rangle$ of prime order, the subgroup $C_G(\psi)$ is a torsion group. Then G has an open normal subgroup which is nilpotent of class at most $h(q)$, where q is the minimal prime divisor of n and $h(q)$ is the Higman function.*

We also study the structure of torsion pro-p groups with FPF automorphisms. E. Zelmanov proved that any torsion pro-p group is locally finite [17]. Combined with results of J. Wilson [16], this implies that any compact torsion group is locally finite. A long-standing problem is whether any compact torsion group has a finite exponent. In [4] W. Herfort reduced this problem to the question of whether any torsion pro-p group has a finite exponent (see also [16]). In this context we shall prove the following theorem.

THEOREM B. *Let p be a prime, and G a torsion pro-p group admitting an FPF automorphism ϕ of finite order. Then G has a finite exponent.*

It was shown in [9] that if a compact torsion group G contains a noncyclic elementary abelian subgroup whose centralizer in G is finite then G is of finite exponent. The above theorem allows us to extend this as follows.

COROLLARY C. *Let a compact torsion group G contain an abelian subgroup whose centralizer in G is finite. Then G is of finite exponent.*

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2. ON LIE ALGEBRAS ADMITTING A FIXED-POINT-FREE AUTOMORPHISM

An automorphism ϕ of a Lie algebra L is called fixed-point-free if $C_L(\phi) = 0$. In this section we shall obtain some sufficient conditions for a Lie algebra with an FPF automorphism to be nilpotent. Although our results hold also for Lie rings, for technical reasons we focus our attention on Lie algebras over a field. So, the term "Lie algebra" will always mean "Lie algebra over a field."

The Higman Theorem says that if a Lie algebra L admits an FPF automorphism of prime order q then L is nilpotent of class at most $h(q)$, where $h(q)$ is the function of q mentioned in the Introduction [6]. A Lie algebra admitting an FPF automorphism of composite order need not be nilpotent. We shall see however that in certain natural situations the existence of an FPF automorphism of finite order implies nilpotency of the Lie algebra.

The following lemma is well-known. We outline the proof for the reader's convenience.

LEMMA 2.1. *Let H be a finite-dimensional Lie algebra of linear transformations of some vector space V (possibly infinite-dimensional). Assume that H is spanned by elements $x_1, x_2, \ldots, x_d$ such that each Lie product in $x_1, x_2, \ldots, x_d$ is nilpotent of index at most m. Then there exists a $\{d, m\}$-bounded number s such that the associative product $h_1 h_2 \cdots h_s$ is zero for any $h_1, h_2, \ldots, h_s \in H$.*

Proof. We can assume that H is of dimension d. Let x be any Lie product in $x_1, x_2, \ldots, x_d$. Since $x^m = 0$, it follows that x is ad-nilpotent in H of index at most $2m - 1$. Arguing as in the proof of Engel's Theorem [7], we arrive at the conclusion that H is nilpotent of $\{d, m\}$-bounded class. Since H is nilpotent, we can choose an ideal D of codimension one in H such that D satisfies the hypothesis of the lemma and $H = D + \langle z \rangle$ for some $z \in \{x_1, x_2, \ldots, x_d\}$. By induction on d we can assume that there exists a $\{d, m\}$-bounded number r such that the associative product $h_1 h_2 \cdots h_r$ is zero for any $h_1, h_2, \ldots, h_r \in D$. It follows that V possesses a series of H-invariant subspaces

$$V = V_1 \geq V_2 \geq \cdots \geq V_{r+1} = 0$$

such that D annihilates each factor-space $W_i = V_i / V_{i+1}$. Since $z^m = 0$, each space W_i possesses a series of z-invariant subspaces

$$W_i = W_{i,1} \geq W_{i,2} \geq \cdots \geq W_{i,m+1} = 0$$

such that z annihilates each factor-space $W_{i,j} / W_{i,j+1}$. Therefore there exists a $\{d, m\}$-bounded number s such that V possesses a series of H-invariant subspaces

$$V = U_1 \geq U_2 \geq \cdots \geq U_{s+1} = 0$$

satisfying the condition that H annihilates each factor-space U_i / U_{i+1}. This completes the proof. ∎

PROPOSITION 2.2. *Let G be a Lie algebra admitting an FPF automorphism ϕ of finite order n. Suppose that there exists a number s such that*

$$[G, \underbrace{C_G(\psi), \ldots, C_G(\psi)}_{s}] = 0$$

for any element ψ of prime order in $\langle\phi\rangle$. Then G is nilpotent of $\{n, s\}$-bounded class.

Proof. Let k be the ground field for G and ω a primitive nth root of 1. Set $L = G \otimes k[\omega]$. We regard L as a Lie algebra over $k[\omega]$ and ϕ as an automorphism of L. If ψ is any element of prime order in $\langle\phi\rangle$ and $H = C_L(\psi)$ then $H = C_G(\psi) \otimes k[\omega]$, whence

$$[L, \underbrace{H, \ldots, H}_{s}] = 0.$$

For any positive integer i we set ${}^iL = \{x \in L; x^\phi = \omega^i x\}$. Assume that k is of characteristic p. Since $C_G(\phi) = 0$, it follows that either $p = 0$ or some p'-subgroup of $\langle\phi\rangle$ is FPF. We therefore can assume that p does not divide n, in which case $L = {}^0L \oplus {}^1L \oplus \cdots \oplus {}^{n-1}L$. The condition $C_G(\phi) = 0$ implies that $C_L(\phi) = 0$, which is the same as ${}^0L = 0$. Also, we note that $[{}^iL, {}^jL] \le {}^{i+j}L$ for any $i, j \in \mathbb{N}$. By Higman's Theorem [6] we can assume that n is not a prime. It is easy to see that ${}^jL$ is contained in the centralizer of some element of prime order from $\langle\phi\rangle$ if and only if $(j, n) \ne 1$. Therefore, by the hypothesis, if $(j, n) \ne 1$ then

$$[L, \underbrace{{}^jL, \ldots, {}^jL}_{s}] = 0.$$

On the other hand, if $(j, n) = 1$, we have

$$[L, \underbrace{{}^jL, \ldots, {}^jL}_{n-1}] = 0.$$

Indeed,

$$\Big[L, \underbrace{{}^jL, \ldots, {}^jL}_{n-1}\Big] = \Big[\sum_i {}^iL, \underbrace{{}^jL, \ldots, {}^jL}_{n-1}\Big] \le \sum_i \Big[{}^iL, \underbrace{{}^jL, \ldots, {}^jL}_{n-1}\Big].$$

Since $(j, n) = 1$, for any i there exists a number $m \le n-1$ such that $i + mj$ is divided by n. Since $[{}^iL, {}^jL] \le {}^{i+j}L$, we observe that

$$[{}^iL, \underbrace{{}^jL, \ldots, {}^jL}_{m}] \le {}^0L = 0$$

which shows that

$$[L, \underbrace{{}^jL, \ldots, {}^jL}_{n-1}] = 0.$$

For definiteness' sake let us assume that $s \ge n-1$, in which case

$$[L, \underbrace{{}^jL, \ldots, {}^jL}_{s}] = 0 \qquad (*)$$

for any j. By the theorem of Kreknin [10] L is soluble of n-bounded derived length; so, by induction on the derived length d of L, we can assume that L' is nilpotent of some $\{n, d\}$-bounded nilpotency class. Combining this with (*), we conclude that the subalgebra $S_j = L' + {}^jL$ is nilpotent of some $\{n, d\}$-bounded nilpotency class, say c, for any j. We observe that S_j is an ideal of L and $L = S_1 + S_2 + \cdots + S_{n-1}$. Therefore L is nilpotent of class at most $c(n-1)$. This is obviously true for G as well. ∎

PROPOSITION 2.3. *Let n, r, t be positive integers, and L an r-generated Lie algebra admitting an FPF automorphism ϕ of order n. Suppose that for any element ψ of prime order in $\langle\phi\rangle$ the subalgebra $C_L(\psi)$ is spanned by a set of elements in which all Lie products are ad-nilpotent in L. Then L is nilpotent.*

If we assume in addition that $C_L(\psi)$ is spanned by a set of elements in which all Lie products are ad-nilpotent in L of index at most t then L is nilpotent of $\{r, n, t\}$-bounded class.

Proof. We shall prove only the second assertion. The first one can be obtained by a similar argument.

Since, by the theorem of Kreknin [10], L is soluble of n-bounded derived length, we can use induction on the derived length d of L. Let M be the last non-trivial term of the derived series of L. By the inductive hypothesis L/M is nilpotent of $\{d, r, n, t\}$-bounded class. Therefore the dimension of L/M is $\{d, r, n, t\}$-bounded. Choose arbitrarily an element $\psi \in \langle\phi\rangle$ of prime order and set $H = C_L(\psi)$. It follows that the dimension of $H/C_H(M)$ is $\{d, r, n, t\}$-bounded. We regard $H/C_H(M)$ as a Lie algebra of linear transformations of M. Lemma 2.1 implies that there exists a $\{d, r, n, t\}$-bounded number s such that

$$[M, \underbrace{H, \ldots, H}_{s}] = 0.$$

Combining this with the fact that L/M is nilpotent of $\{d, r, n, t\}$-bounded class, we conclude that there exists a $\{d, r, n, t\}$-bounded number v such that

$$[L, \underbrace{H, \ldots, H}_{v}] = 0.$$

The result is now immediate from Proposition 2.2. ∎

3. PROOF OF THEOREM A

The following lemma is helpful and will be used frequently without explicit references (see [14, Lemma 3.2] or [4, Lemma 2] for the proof).

LEMMA 3.1. *Let ϕ be an automorphism of finite order of a profinite group G. Assume that any open subgroup of G has index prime to the order of ϕ and let N be a ϕ-invariant closed normal subgroup of G. Then $C_{G/N}(\phi) = C_G(\phi)N/N$.*

THEOREM A. *Let p be a prime, and G a finitely generated pro-p group admitting an FPF automorphism ϕ of finite order n relatively prime to p. Assume that for any element $\psi \in \langle\phi\rangle$ of prime order, the subgroup $C_G(\psi)$ is a torsion group. Then G has an open normal subgroup which is nilpotent of class at most $h(q)$, where q is the minimal prime divisor of n and $h(q)$ is the Higman function.*

Proof. For any i we let D_i denote the ith dimension subgroup of G in characteristic p. Set $L_i = D_i/D_{i+1}$ and denote by $DL(G)$ the Lie algebra over the field with p elements corresponding to the series $\{D_i\}$ (see for example [13] for details). Let L be the subalgebra of $DL(G)$ generated by L_1. We view ϕ as an automorphism of L. If $x \in D_i \backslash D_{i+1}$ and if x has finite order then, by a result of M. Lazard [11, 6.8], xD_{i+1} is ad-nilpotent in $DL(G)$ of index at most the order of x. Combining this with Lemma 3.1 and with the hypothesis that $C_G(\psi)$ is torsion for any element ψ of prime order in $\langle\phi\rangle$, we derive that $C_L(\psi)$ is spanned by elements in which every Lie product is ad-nilpotent in $DL(G)$. Lemma 3.1 also implies that $C_L(\phi) = 0$. Since L is finitely generated, Proposition 2.3 shows that L is nilpotent. The theory of analytic p-adic groups says that the nilpotency of L is equivalent to the existence of an open characteristic powerful subgroup P in G (see [13, Theorem 4.6] and [2, Chap. 3]). The subgroup P is again finitely generated and so the elements of finite order of P form a finite subgroup T [2, Theorem 4.20]. Let us choose an open normal ϕ-invariant subgroup H of G such that $H \leq P$ and $H \cap T = 1$. If $\nu \in \langle\phi\rangle$ is of order q then ν induces on H a fixed-point-free automorphism of order dividing q. Combining the theorem of G. Higman [6] with Lemma 3.1 (see [14, 3.3] for detail), we conclude that H is nilpotent of class at most $h(q)$. The proof is complete. ∎

4. CENTRALIZERS IN COMPACT TORSION GROUPS

The aim of this section is to establish Theorem B and Corollary C.

LEMMA 4.1. *Let h be a positive integer, and G a nilpotent group of class at most h. Let $x \in G$ be an element of order dividing p^n. Then x commutes with $y^{p^{n(h-1)}}$ for any $y \in G$.*

Proof. Set $z = y^{p^{n(h-2)}}$. We can assume that $G = \langle x, y\rangle$. By induction on h we also assume that $[x, z] \in \gamma_h(G)$. Since $z \in Z_2(G)$, we write

$$1 = \left[x^{p^n}, z\right] = \left[x, z^{p^n}\right] = \left[x, y^{p^{n(h-1)}}\right].$$

∎

It is well-known that any soluble compact torsion group is of finite exponent (see for example [12, Proposition 1]). Since any soluble subgroup of a compact group is contained in a closed soluble subgroup, we obtain the following lemma.

LEMMA 4.2. *Let G be a compact torsion group. Then every soluble subgroup of G has finite exponent.*

LEMMA 4.3. *Let n, h be positive integers, and G a torsion pro-p group. Assume that for any $x, y \in G$ the group $\langle x, y\rangle$ contains a normal subgroup H of nilpotency class at most h such that $\langle x, y\rangle / H$ has exponent at most p^n. Then G has finite exponent.*

Proof. Assume that G is a counterexample to the lemma. Then the set $S = \{x^{p^n}; x \in G\}$ has elements of order p^k for any natural k. Clearly, we can choose countably many elements $x_1, x_2, \ldots \in S$ such that

(1) x_i is of order p^i, $i = 1, 2, \ldots$.

(2) For any i there exists $y_i \in S$ such that $x_{i+1} = y_i^{p^{i(h-1)}}$.

Then, by the hypothesis, any two elements x_i, y_j generate a subgroup of class at most h and therefore, by Lemma 4.1, x_i and x_j commute. It follows that the subgroup $\langle x_1, x_2, \ldots\rangle$ is abelian. Since $\langle x_1, x_2, \ldots\rangle$ is not of finite exponent, we obtain a contradiction with Lemma 4.2. ∎

A finite group G is said to be of rank r if any subgroup of G is generated by at most r elements. The following lemma is necessary for the proof of Theorem B.

LEMMA 4.4. *Let G be a finite p-group of rank r and exponent t. Then the order of G is $\{r, t\}$-bounded.*

Proof. By Theorem 2.13 of [2] G has a powerful characteristic subgroup N of index at most $p^{\mu(r)}$, where $\mu(r)$ is a number depending only on r. Corollary 2.8 in [2] shows that N is a product of at most r cyclic subgroups. Therefore N is of order at most t^r and the lemma follows. ∎

THEOREM B. *Let p be a prime and G a torsion pro-p group admitting an FPF automorphism ϕ of finite order. Then G has finite exponent.*

Proof. By Zelmanov's Theorem G is locally finite [17]. Of course, we can assume that $\phi \neq 1$. Since ϕ is FPF, it follows that so is the maximal p'-subgroup of $\langle \phi \rangle$ and therefore we can assume that p does not divide the order n of ϕ. If ϕ is of prime order then, by Higman's Theorem [6], G is nilpotent and the result follows from Lemma 4.2. Thus, assume that n is not a prime. If $\psi \in \langle \phi \rangle$ is any element of order less than n then the centralizer $C_G(\psi)$ is a closed subgroup admitting an FPF automorphism of order less than n. Hence, arguing by induction on n, we can assume that $C_G(\psi)$ is of finite exponent, say t, for any $\psi \in \langle \phi \rangle$ of order less than n. Let now P be any ϕ-invariant d-generated subgroup of G. Let D_i denote the ith dimension subgroup of P, $L_i = D_i/D_{i+1}$. As in the proof of Theorem A we consider the Lie algebra $DL(G) = \oplus L_i$ and write L for the subalgebra of $DL(G)$ generated by L_1. We view ϕ as an automorphism of L. The hypothesis $C_G(\phi) = 1$ implies that $C_L(\phi) = 0$. As was shown by M. Lazard, if $x \in D_i \backslash D_{i+1}$ is of order dividing t then xD_{i+1} is ad-nilpotent in $DL(G)$ of index at most t [11, 6.8]. Therefore for any $\psi \in \langle \phi \rangle$ of order less than n the centralizer $C_L(\psi)$ is spanned by elements in which every Lie product is ad-nilpotent of index at most t. Proposition 2.3 now tells us that L is nilpotent of $\{n, d, t\}$-bounded class, say c. Note that t depends only on the maximum of the exponents of $C_G(\psi)$ and not on the choice of the subgroup P. Since L is nilpotent of class at most c, it follows that P has a characteristic powerful subgroup H of $\{p, d, c\}$-bounded index (see [9, Proposition 1]). So the index of H is bounded in terms of the numerical parameters p, n, d, t which do not depend on the choice of P. Since P is d-generated, we conclude that the minimal number of generators of H is also $\{p, n, d, t\}$-bounded. Let us denote this number by r. By Lemma 2.9 of [2], H is of rank r. Let $\nu \in \langle \phi \rangle$ be of prime order q and let $m = |C_P(\nu)|$. It follows (Lemma 4.4) that the order of $C_H(\nu)$—and hence m—is $\{p, n, d, t\}$-bounded. A theorem of Khukhro says that any finite group admitting an automorphism of order q with at most m fixed points has a normal nilpotent subgroup of q-bounded nilpotency class and $\{q, m\}$-bounded index [8]. Applying this result we conclude that P has a normal subgroup of q-bounded nilpotency class and $\{q, m\}$-bounded index.

Now let x, y be arbitrary elements of G. The subgroup $\langle x, y \rangle$ is contained in a ϕ-invariant d-generated subgroup P of G, where $d \leq 2n$. Therefore $\langle x, y \rangle$ has a normal subgroup of q-bounded nilpotency class and $\{q, m\}$-bounded index. The result now follows from Lemma 4.3. ∎

COROLLARY C. *Assume that a compact torsion group G contains an abelian subgroup Q whose centralizer in G is finite. Then G is of finite exponent.*

Proof. It is well-known that G is profinite [5, 28.20] so we can choose a normal open subgroup H such that $H \cap C_G(Q) = 1$. By virtue of Wilson's

Theorem [16], H has a finite series of characteristic closed subgroups

$$H = H_1 \geq H_2 \geq \cdots \geq H_s = 1,$$

all of whose quotients are either of finite exponent or pro-p groups for some primes p. It is sufficient to show that any such pro-p group H_i/H_{i+1} has finite exponent. We therefore can assume that H is a pro-p group. Note that Q is finite because so is $C_G(Q)$. The group Q acts on H by inner automorphisms and this action is fixed-point-free. We remark that since G is locally finite [17], it follows that the maximal p'-subgroup of Q acts on H fixed-point-freely. Therefore without any loss of generality we can assume that Q has no elements of order p. If R is any non-trivial proper subgroup of Q then $C_H(R)$ is a closed Q-invariant subgroup admitting a fixed-point-free action of a finite abelian group whose order is strictly less than that of Q. So, arguing by induction on $|Q|$, we can assume that $C_H(R)$ is of finite exponent for any non-trivial proper subgroup R of Q. If Q is not cyclic then it contains a non-cyclic elementary subgroup A. Since $C_H(\alpha)$ has finite exponent for any $\alpha \in A\backslash\{1\}$, by [9] the group H is of finite exponent and we are done. We therefore can assume that Q is cyclic. The claim is now immediate from Theorem B. ∎

REFERENCES

1. Th. Berger, Nilpotent fixed point free automorphism groups of solvable groups, *Math. Z*. **131** (1973), 305–312.
2. J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal, "Analytic p-adic Groups," Cambridge Univ. Press, Cambridge, UK, 1991.
3. D. Gorenstein, "*Finite Simple Groups:* An Introduction to Their Classification," Plenum, New York/London, 1982.
4. W. Herfort, Compact torsion groups and finite exponent, *Arch. Math*. **33** (1979), 404–410.
5. E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis," Vol. 2, Springer, New York/Berlin, 1970.
6. G. Higman, Groups and rings which have automorphisms without nontrivial fixed elements, *J. London Math. Soc. Ser*. *(2)* **32** (1957), 321–334.
7. N. Jacobson, "Lie algebras," Wiley–Interscience, New York, 1962.
8. E. I. Khukhro, Groups and Lie rings admitting almost regular automorphisms of prime order, *in*: "Proc. International Conference on the Theory of Groups, Suppl." *Rend. Circ. Mat. Palermo* (1990), 183–192.
9. E. I. Khukhro and P. Shumyatsky, Bounding the exponent of a finite group with automorphisms, *J. Algebra* **212** (1999), 363–374.
10. V. A. Kreknin, The solubility of Lie algebras with regular automorphisms of finite period, *Soviet Math. Dokl*. **4** (1963), 683–685.
11. M. Lazard, Groupes analytiques p-adiques, *Inst. Hautes Etudes Sci. Publ. Math*., **26** (1965), 389–603.

12. J. R. McMullen, Compact torsion groups, *in* "Proc. 2nd International Conference on the Theory of Groups, Canberra, 1973," Lecture Notes in Math., Vol. **372**, Springer-Verlag, New York/Berlin 1974.
13. A. Shalev, Finite p-groups, *in* "Finite and Locally Finite Groups," NATO ASI Series, Vol. 471, pp. 401–450, Kluwer, Dordrecht, 1995.
14. P. Shumyatsky, Centralizers in groups with finiteness conditions, *J. Group Theory* **1** (1998), 275–282.
15. J. G. Thompson, Finite groups with fixed-point-free automorphisms of prime order, *Proc. Natl. Acad. Sci. U.S.A*. **45** (1959), 578–581.
16. J. S. Wilson, On the structure of compact torsion groups, *Monatsh. Math*, **96** (1983), 57–66.
17. E. Zelmanov, On periodic compact groups, *Israel J. Math*. **77** (1992), 83–95.